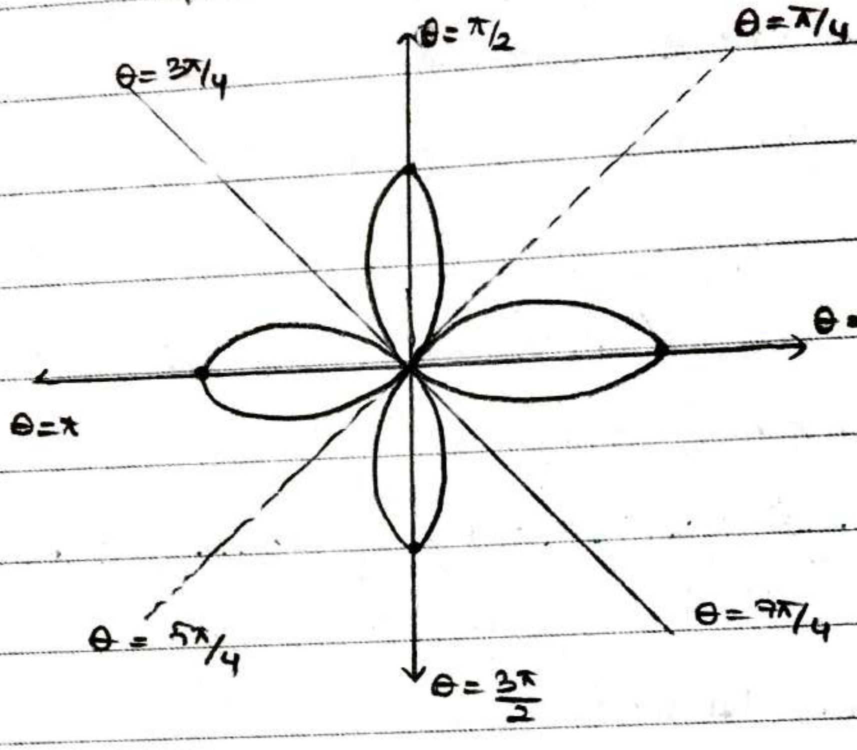
* **Ex:** (11.4 / 20)

Graph $r = \cos 2\theta$



$\theta = \pi \rightarrow$

$\theta = \pi/2 \rightarrow$

CHAPTER 14 - PARTIAL DERIVATIVES

14.1 → FUNCTIONS OF SEVERAL VARIABLES

Up to this point

$y = f(x)$

Now

output

- $y = f(x_1, x_2, x_3, \dots x_n)$ multivariable function

input

- $z = f(x, y)$ function with two variables.

ex: $z = \sqrt{y - x^2}$

Domain? $\longrightarrow$ $y \geq x^2$

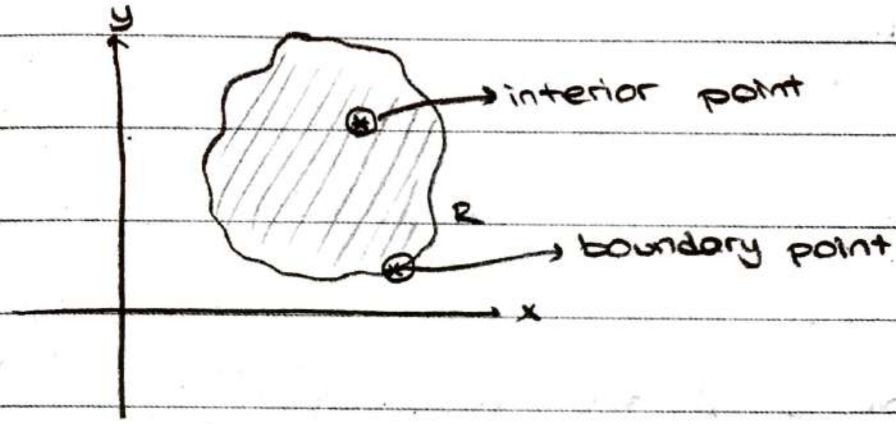
Range? $\longrightarrow$ $[0, \infty)$

ex: $w = \dfrac{1}{x^2 + y^2 + z^2}$

Domain? $\longrightarrow$ $(x, y, z) \neq (0, 0, 0)$

Range? $\longrightarrow$ $(0, 0, 0)$

Functions of Two Variables



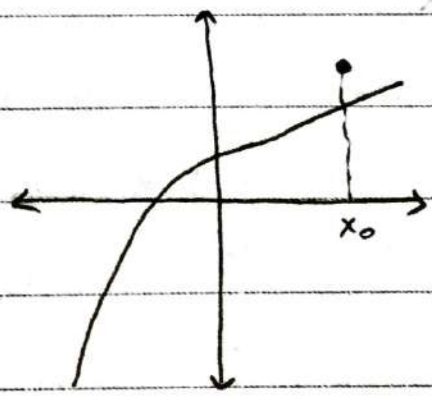
14.2 → LIMITS AND CONTINUITY IN HIGHER DIMENSIONS

Definition: (informal definition of limit)

we say $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if $f(x,y)$ is close to L for every (x,y) close to (x_0,y_0) (except $(x,y) = (x_0,y_0)$)

$\lim_{x\to x_0} f(x) = L$



ex: $\lim_{(x,y)\to(1,0)} x+y = 1$

$\lim_{(x,y)\to(2,5)} e^{x^2 y} = 20$

* Properties

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L+M$

2. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^n = L^n$ n: positive integer

3. $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$ if this is defined

ex: $$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \frac{0-0+3}{0+0-1} = -3$$

Sandwich Theorem

$f(x,y) \le g(x,y) \le h(x,y)$ for all x,y

if $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L$

Then $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L$

ex: $$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = ?$$

$$0 \le \left|\frac{4xy^2}{x^2+y^2}\right| = |4x|\left|\frac{y^2}{x^2+y^2}\right| \le |4x|$$

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |4x|$$

$$\lim_{(x,y)\to(0,0)} \left|\frac{4xy^2}{x^2+y^2}\right| = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

Lemma $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$$

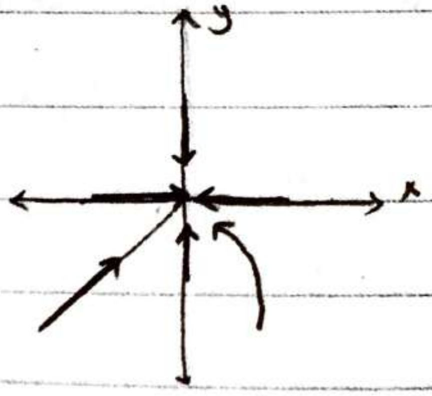
Proof $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$

$$\lim_{(x,y)\to(x_0,y_0)} -|f(x,y)| = \lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0$$

- By sandwich teorem

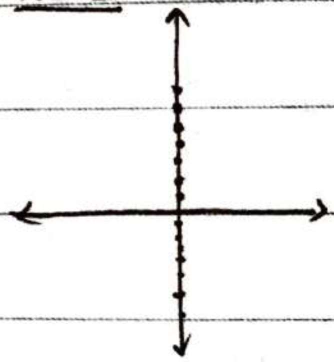
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$$

ex: $\lim_{(x,y)\to(0,0)} \left(\frac{x}{y}\right) = ?$



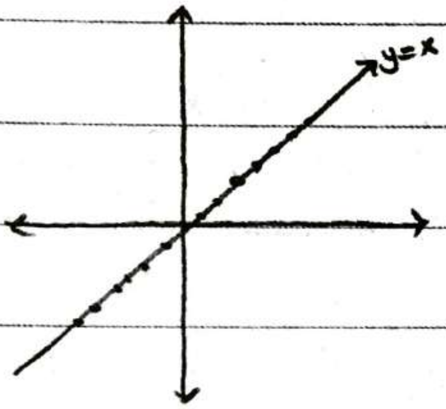
continue →

$\underline{x=0}$



$$\lim_{(x,y)\to(0,0)} \frac{x}{y} = \lim_{y\to 0} \frac{0}{y} = \lim_{y\to 0} 0 = 0$$

$\underline{x=y}$



$$\lim_{(x,y)\to(0,0)} \frac{x}{y} = \lim_{y\to 0} \frac{y}{y} = \lim_{y\to 0} 1 = 1$$

- Limit does not exist because it depends on the path taken to approach $(0,0)$

Continuity

A function $f(x,y)$ is continuous at the point (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to x_0,y_0} f(x,y) = f(x_0,y_0)$

ex: Is

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$ = ?

$\boxed{y=0}$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

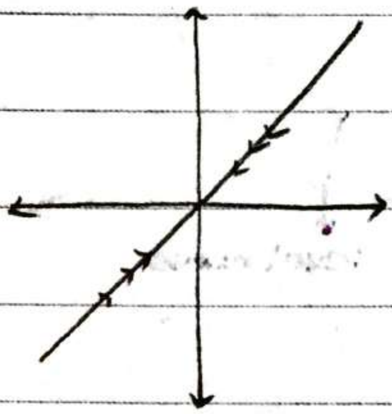
$\boxed{y=x}$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{2xx}{x^2+x^2} = \lim\limits_{x\to 0} 1 = 1$

Limit does not exist.

First Method

$\boxed{y=kx}$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \dfrac{2xkx}{x^2+(kx)^2} = \lim\limits_{x\to 0} 2k \dfrac{x^2}{x^2+k^2x^2}$

$= \lim\limits_{x\to 0} \dfrac{2k}{1+k^2} = \dfrac{2k}{1+k^2}$



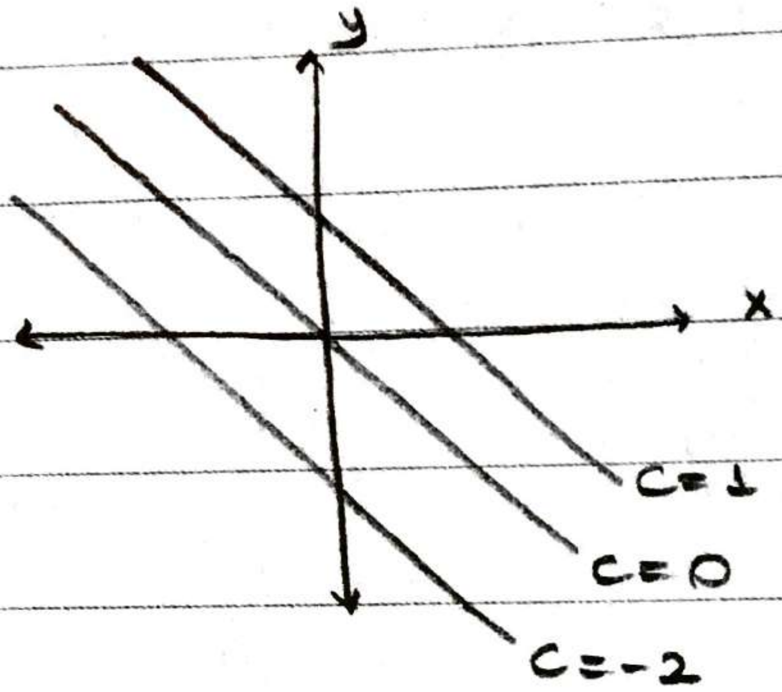
Since limit depends on k, limit does not exist

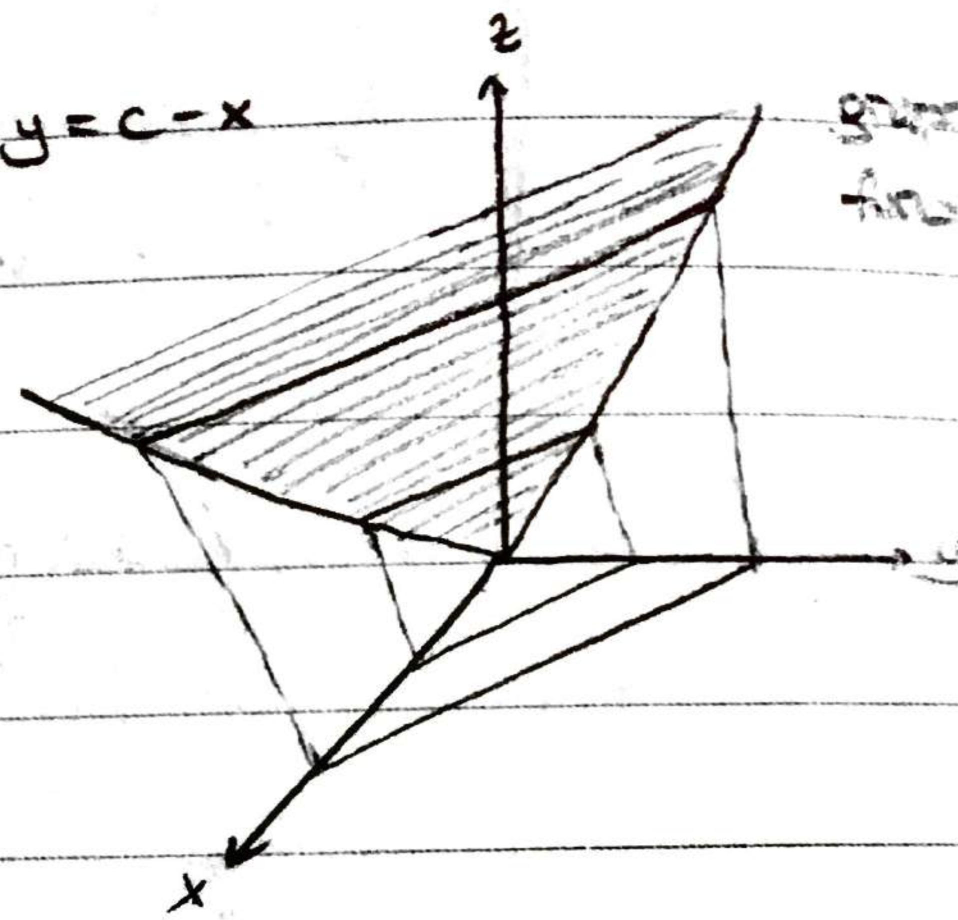
Second Method

Definition: The set of points in the plane where $f(x,y) = \text{constant}$ is called a level curve of f.

ex: $f(x,y) = x + y$

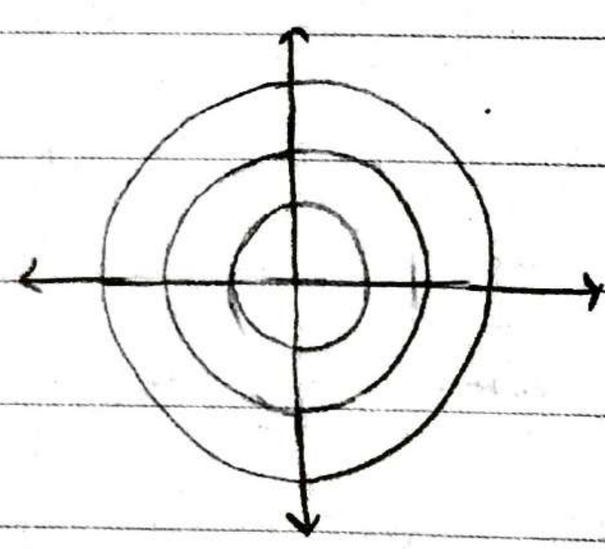
$x+y = \text{constant} = c \Rightarrow y = c - x$

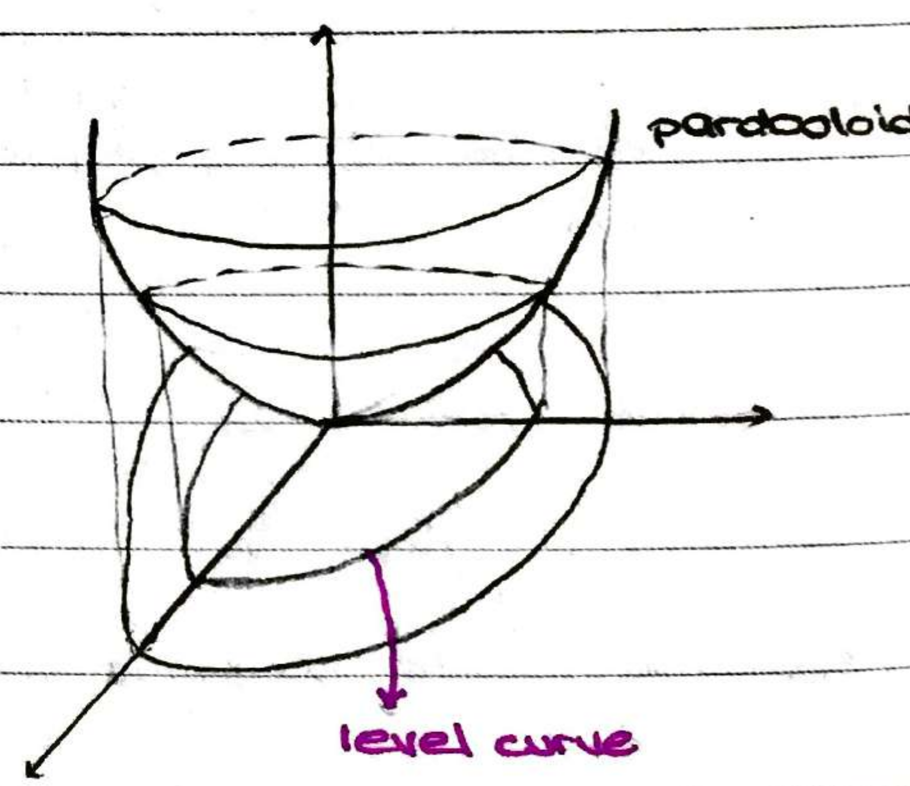




ex: $f(x,y) = x^2 + y^2$

$x^2 + y^2 = c$





book (9)

ex:

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy - y - 2x + 2}{x - 1} \quad \frac{0}{0}$$

$$= \lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{y(x-1) - 2(x-1)}{x-1}$$

$$= \lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} (y-2) = -1$$

(book 61)
ex: $\lim_{(x,y)\to(0,0)} \left(\frac{x^3-xy^2}{x^2+y^2}\right)$

$$\lim_{(x,y)\to(0,0)} \frac{x^3-k^2x^3}{x^2+k^2x^2} = x\left(\frac{1-k^2}{1+k^2}\right)$$

$\boxed{y=kx}$

$$= \lim_{x\to 0}\left(\frac{1-k^2}{1+k^2}\right)x = 0$$

this says if limit exists it must be zero.

Lets show that limit exists,

$$0 \le \left|\frac{x^3-xy^2}{x^2+y^2}\right| = |x| \cdot \left|\frac{x^2-y^2}{x^2+y^2}\right| \le |x| \cdot 1$$

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |x| = 0$$

By sandwich theorem $\lim_{(x,y)\to(0,0)} \left|\frac{x^3-xy^2}{x^2+y^2}\right| = 0$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = 0$$

Paths to approach $(0,0)$

- ✓ $x=0$
- ✓ $y=0$
- ✓ $y=x$
- ✓ $y=kx$
- ✓ $y=kx^2$

(book 48)
ex: $\lim_{(x,y)\to(0,0)} \left(\frac{x^2y}{x^4+y^2} \right)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2kx}{x^4+k^2x^2}$$

$\boxed{y=kx}$

$$= \lim_{x\to 0} x\frac{k}{x^2+k^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2kx^2}{x^4+k^2x^4}$$

$\boxed{y=kx^2}$

$$= \lim_{x\to 0} \frac{k}{1+k^2} = \underbrace{\frac{k}{1+k^2}}$$

limit depends on k hence does not exist.

14.3 → PARTIAL DERIVATIVES

$f(x,y)$

"del" $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial x} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

ex: $f(x,y) = x^2y$. Find $\frac{\partial f}{\partial x}(1,2)$

$$\frac{\partial f}{\partial x}(1,2) = \lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2 \cdot 2 - 1 \cdot 2}{h} = \lim_{h\to 0} \frac{(1+2h+h^2)\cdot 2-2}{h}$$

$$= \lim_{h\to 0} \frac{4h+2h^2}{h} = \lim 4+2h = 4$$

→ Alternatively

$$\frac{\partial f}{\partial x} = 2xy \quad \Rightarrow \quad \frac{\partial f}{\partial x}(1,2) = 2.1.2 = 4$$

ex: $f(x,y) = e^y \sin(xy)$

$$\frac{\partial f}{\partial x} = ?$$

$$\frac{\partial f}{\partial x} = e^y \sin(xy) = e^y (\cos(xy))\, x$$

ex: Find $\frac{\partial z}{\partial x}$ if $yz - \ln z = x + y$

where z is a function of x and y

$$\frac{\partial}{\partial x}\left(yz - \ln z\right) = \frac{\partial}{\partial x}\left(x + \underline{y}\right)$$

$$y \cdot \frac{\partial z}{\partial x} - \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}}$$

ex: $f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence f is not continuous at $(0,0)$

$$\frac{\partial f}{\partial x}(0.0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0.0)}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = 0$$

$$\frac{\partial f}{\partial x}(0.0) = \lim_{k \to 0} \frac{f(0, 0+k) + f(0.0)}{k} = 0$$

Thus both partial derivatives $\frac{\partial f}{\partial x}(0.0)$ exist but f is not continuous at (0.0).